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OPTIMAL LYAPUNOV INEQUALITIES FOR BOUNDARY VALUE PROBLEMS

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Abstract. This work is devoted to review some recent results on L_p Lyapunov-type inequalities ($1 \leqslant p \leqslant \infty$) for resonant differential equations. In the case of Ordinary Differential Equations, we consider Neumann boundary conditions and an explicit optimal result is obtained. Moreover, it is also treated the case in which the resonance appears at higher eigenvalues. We also study mixed boundary conditions. From this study, and under some natural restrictions on the linear coefficient, the relation between Neumann boundary conditions and disfocality arises in a natural way. For Partial Differential Equations it is proved that the relation between the quantities p and $N/2$ plays a crucial role in order to obtain L_p Lyapunov-type inequalities, for resonant linear problems with Neumann boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^N$. This fact shows a deep difference with respect to the ordinary case. Combining these linear results with Schauder fixed point theorem, we can obtain some new results about the existence and uniqueness of solutions for resonant nonlinear problems. Finally, we comment some conclusions on systems of equations.

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