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FURTHER EXTENSION OF FURUTA INEQUALITY

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Abstract. If $A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 \geqslant B \geqslant 0$, with $A_1 > 0$, $t_1, t_2, \cdots, t_{n-1}$, $t_n \in [0,1]$ and $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Then the following inequality holds for $r \geqslant t_n$

$$\begin{aligned} {A_{2n}}^{1-t_n+r} \geqslant \{ & {A_{2n}}^{\frac{r}{2}} [{A_{2n-1}}^{\frac{-t_n}{2}} \{{A_{2(n-1)}}^{\frac{t_{n-1}}{2}} \cdots {A_4}^{\frac{t_2}{2}} [{A_3}^{\frac{-t_2}{2}} \{{A_2}^{\frac{t_1}{2}} ({A_1}^{\frac{-t_1}{2}} B^{p_1} {A_1}^{\frac{-t_1}{2}})^{p_2} \\ & {A_2}^{\frac{t_1}{2}} \}^{p_3} {A_3}^{\frac{-t_2}{2}}]^{p_4} {A_4}^{\frac{t_2}{2}} \cdots {A_{2(n-1)}}^{\frac{t_{n-1}}{2}} \}^{p_{2n-1}} {A_{2n-1}}^{\frac{-t_n}{2}}]^{p_{2n}} {A_{2n}}^{\frac{r}{2}} \}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}}, \end{aligned}$$

where $\eth[2n] = \{\cdots [\{[(p_1 - t_1)p_2 + t_1]p_3 - t_2\}p_4 + t_2]p_5 - \cdots - t_n\}p_{2n} + t_n$.

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