Journal of Mathematical Inequalities
Volume 4, Number 4 (2010), 481–503

SOME NEW REFINED HARDY–TYPE INEQUALITIES WITH KERNELS

ALEKSANDRA ČIŽMEŠIJA, KRISTINA KRULIĆ AND JOSIP PEČARIĆ

Abstract. By using the notion of the subdifferential of a convex function, we state and prove a new general refined weighted Hardy-type inequality for convex functions and the integral operator with a non-negative kernel. We point out that the obtained result generalizes and refines the classical one-dimensional Hardy's, Pólya–Knopp's, and Hardy–Hilbert's inequalities, as well as related dual inequalities. We show that our results may be seen as generalizations of some recent results related to Riemann-Liouville's and Weyl's operator, as well as a generalization and a refinement of the so-called Godunova's inequality.

Mathematics subject classification (2010): 26D10, 26D15.
Keywords and phrases: Hardy's inequality, Hardy-Hilbert's inequality, weights, power weights, convex functions, Hardy's integral operator, kernel.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (EDS.), *Handbook of mathematical functions with formulas, graphs and mathematical tables*, 9^{th} printing, Dover, New York, 1972.
[2] K. F. ANDERSEN, *Weighted inequalities for the Stieltjes transformation and Hilbert's double series*, Proc. Roy. Soc. Edinburgh Sect. A **86(1-2)** (1980), 75–84.
[3] R. P. BOAS, *Some integral inequalities related to Hardy's inequality*, J. Anal. Math. **23** (1970), 53–63.
[4] A. ČIŽMEŠIJA, S. HUSSAIN, AND J. PEČARIĆ, *Some new refinements of Hardy and Pólya–Knopp's inequalities*, J. Function Spaces Appl, **7(2)** (2009), 167–186.
[5] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Mixed means and Hardy's inequality*, Math. Inequal. Appl. **1(4)** (1998), 491–506.
[6] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Classical Hardy's and Carleman's inequalities and mixed means*, in: T. M. Rassias (ed.), *Survey on Classical Inequalities*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2000, 27–65.
[7] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Some new generalizations of inequalities of Hardy and Levin-Cochran-Lee*, Bull. Austral. Math. Soc. **63(1)** (2001), 105–113.
[8] A. ČIŽMEŠIJA AND J. PEČARIĆ, *On Bicheng-Debnath's generalizations of Hardy's integral inequality*, Int. J. Math. Math. Sci. **27(4)** (2001), 237–250.
[9] A. ČIŽMEŠIJA, J. PEČARIĆ, AND L.-E. PERSSON, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory **125** (2003), 74–84.
[10] W. D. EVANS, A. GOGATISHVILI, AND B. OPIC, *The ρ-quasiconcave functions and weighted inequalities*, in: C. Bandle, A. Gilány, L. Losonczi, Z. Páles, and M. Plum (eds.), *Inequalities and Applications. Conference on Inequalities and Applications, Noszvaj (Hungary), September 2007*, International Series of Numerical Mathematics, Vol. 157, 121–132, Birkhäuser, Basel/Boston/Berlin, 2008.
[11] E. K. GODUNOVA, *Generalization of a two-parameter Hilbert inequality* (in Russian), Izv. Vyss. Učebn. Zaved. Matematika 1967 **54(1)** (1967), 35–39.
[12] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, 2^{nd} edition, Cambridge University Press, Cambridge, 1967.

Paper JMI-04-44

[13] H. P. HEINIG, *A note on Hardy's inequality in Orlitz spaces*, in: *Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992)*, 185–194, Stud. Adv. Math., CRC, Boca Raton, FL, USA, 1995.
[14] M. JOHANSSON, L.-E. PERSSON, AND A. WEDESTIG, *Carleman's inequality: history, proofs and some new generalizations*, JIPAM J. Inequal. Appl. Math. **3(4)** (2003), 19 pages, electronic.
[15] S. KAIJSER, L.-E. PERSSON, AND A. ÖBERG, *On Carleman and Knopp's Inequalities*, J. Approx. Theory **117** (2002), 140–151.
[16] S. KAIJSER, L. NIKOLOVA, L-E. PERSSON, AND A. WEDESTIG, *Hardy type inequalities via convexity,* Math. Inequal. Appl. **8(3)** (2005), 403–417.
[17] R. KERMAN, *Construction of weights in modular inequalities involving operators of Hardy type*, in: H. Hudzik, L. Skrzypczak (eds.), *Function spaces. The fifth conference: Proceedings of the conference at Poznań, Poland, 1998*, Lecture Notes in Pure and Appl. Math. 213, 269–278, Marcel Dekker, New York/Basel, 2000.
[18] K. KRULIĆ, J. PEČARIĆ, AND L.-E. PERSSON, *A new Hardy–type inequality with kernel*, Math. Inequal. Appl.**12(3)** (2009), 473–485.
[19] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The prehistory of the Hardy inequality*, Amer. Math. Monthly **113** (2006), 715–732.
[20] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The Hardy inequality - about its history and some related results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[21] A. KUFNER AND L.-E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co, Singapore/New Jersey/London/Hong Kong, 2003.
[22] Q. LAI, *Weighted modular inequalities for Hardy type operators*, Proc. London Math. Soc. (3) **79(3)** (1999), 649–672.
[23] N. LEVINSON, *Generalizations of an inequality of Hardy*, Duke Math. J. **31** (1964), 389–394.
[24] C. NICULESCU AND L.-E. PERSSON, *Convex functions and their applications. A contemporary approach*, CMC Books in Mathematics, Springer, New York, 2006.
[25] J. A. OGUNTUASE, L.-E. PERSSON, AND A. ČIŽMEŠIJA, *Multidimensional Hardy-type Inequalities via convexity*, Bull. Austral. Math. Soc. **77** (2008), 245–260.
[26] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Academic Press, San Diego, 1992.
[27] G. SINNAMON, *A note on the Stieltjes transformation*, Proc. Roy. Soc. Edinburgh Sect A **110** (1988), 73–78.
[28] B. YANG AND T. M. RASSIAS, *On the way of weight coefficents and research for the Hilbert–type inequalities*, Math. Inequal. Appl. **6(4)** (2003), 625–658.