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OPERATOR FUNCTIONS ON CHAOTIC ORDER INVOLVING ORDER PRESERVING OPERATOR INEQUALITIES

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Abstract. An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all vectors x in a Hilbert space, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. Let $\log A \geqslant \log B$ and $r_1, r_2, \dots, r_n \geqslant 0$ and any fixed $\delta \geqslant 0$, and

$$p_1 \geqslant \delta, \quad p_2 \geqslant \frac{\delta + r_1}{p_1 + r_1}, \; \dots, \; p_k \geqslant \frac{\delta + r_1 + r_2 + \dots + r_{k-1}}{\mathfrak{q}[k-1]}, \; \dots, \; p_n \geqslant \frac{\delta + r_1 + r_2 + \dots + r_{n-1}}{\mathfrak{q}[n-1]}.$$

Let $\mathfrak{F}_n(p_n, r_n)$ be defined by

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta + r_1 + r_2 + \dots + r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}}.$$

Then the following inequalities (i), (ii) and (iii) hold:

(i) $A^{\frac{p_{k-1}}{2}} \mathfrak{F}_{k-1}(p_{k-1}, r_{k-1}) A^{\frac{p_{k-1}}{2}} \geqslant \mathfrak{F}_k(p_k, r_k)$ for k such that $1 \leqslant k \leqslant n$,

(ii)
$$\begin{aligned} B^{\delta} &\geqslant A^{\frac{-r_1}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{\delta + r_1}{p_1 + r_1}} A^{\frac{-r_1}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2)}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{\frac{\delta + r_1 + r_2}{(p_1+r_1)p_2 + r_2}} A^{\frac{-(r_1+r_2)}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}} \{A^{\frac{r_3}{2}} [A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}]^{p_3} A^{\frac{r_3}{2}}\}^{\frac{\delta + r_1 + r_2 + r_3}{\{(p_1+r_1)p_2 + r_2\}p_3 + r_3}} A^{\frac{-(r_1+r_2+r_3)}{2}} \\ &\;\;\vdots \\ &\geqslant A^{\frac{-(r_1+r_2+\dots+r_n)}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta + r_1 + r_2 + \dots + r_n}{\mathfrak{q}[n]}} A^{\frac{-(r_1+r_2+\dots+r_n)}{2}}, \end{aligned}$$

(iii) $\mathfrak{F}_n(p_n, r_n)$ is a decreasing function of both $r_n \geqslant 0$ and $p_n \geqslant \frac{\delta + r_1 + r_2 + \dots + r_{n-1}}{\mathfrak{q}[n-1]}$, where $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$ are defined as follows:

$$\mathbb{C}_{A,B}[n] = A^{\frac{r_n}{2}} \left\{ A^{\frac{r_{n-1}}{2}} [\dots A^{\frac{r_3}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{p_3} A^{\frac{r_3}{2}} \dots]^{p_{n-1}} A^{\frac{r_{n-1}}{2}} \right\}^{p_n} A^{\frac{r_n}{2}}$$

and

$$\mathfrak{q}[n] = [\dots\{(p_1 + r_1)p_2 + r_2\}p_3 + \dots r_{n-1}]p_n + r_n.$$

We remark that (ii) can be considered as *"a satellite inequality to chaotic order"*.

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