OPERATOR FUNCTIONS ON CHAOTIC ORDER INVOLVING ORDER PRESERVING OPERATOR INEQUALITIES

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Abstract. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(T x, x) \geq 0$ for all vectors $x$ in a Hilbert space, and $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible. Let $\log A \geq \log B$ and $r_1, r_2, ..., r_n \geq 0$ and any fixed $\delta \geq 0$, and

\[ p_1 \geq \delta, \quad p_2 \geq \frac{\delta + r_1}{p_1 + r_1}, \quad ..., \quad p_k \geq \frac{\delta + r_1 + r_2 + ... + r_{k-1}}{q[k-1]}, \quad ..., \quad p_n \geq \frac{\delta + r_1 + r_2 + ... + r_{n-1}}{q[n-1]} . \]

Let $\delta_n(p_n, r_n)$ be defined by

\[ \delta_n(p_n, r_n) = A^{-\frac{n}{2}} C_{A,B}[n] \frac{\delta^{\frac{1}{2}} + \rho_{(p_2+r_1)/(p_2+p_1)} A^{\frac{1}{2}}}{q[n]} A^{-\frac{1}{2}} . \]

Then the following inequalities (i), (ii) and (iii) hold:

(i) $A^{\frac{p_k-1}{2}} \delta_k - (p_{k-1}, r_{k-1}) A^{\frac{p_k-1}{2}} \geq \delta_k(p_k, r_k)$ for $k$ such that $1 \leq k \leq n$,

(ii) $B^\delta \geq A^{-\frac{\delta}{2}} \left( A^{\frac{p_1}{2}} B^{p_1} A^{\frac{1}{2}} \right)^{\frac{\delta}{p_1}} A^{-\frac{\delta}{2}}$

\[ \geq A^{-\frac{(r_1 + r_3)}{2}} \left( A^{\frac{p_1}{2}} \left( A^{\frac{p_1}{2}} B^{p_1} A^{\frac{1}{2}} \right)^{\frac{p_2}{2} A^{\frac{1}{2}}} \left( r_1 + r_3 \right) \right) A^{-\frac{(r_1 + r_3)}{2}} \]

\[ \geq A^{-\frac{(r_1 + r_3 + r_6)}{2}} \left( A^{\frac{p_1}{2}} \left( A^{\frac{p_1}{2}} B^{p_1} A^{\frac{1}{2}} \right)^{\frac{p_2}{2} A^{\frac{1}{2}}} \left( r_1 + r_3 + r_6 \right) \right) A^{-\frac{(r_1 + r_3 + r_6)}{2}} \]

\[ \quad \vdots \]

\[ \geq A^{-\frac{(r_1 + r_3 + ... + r_n)}{2}} \frac{\delta^{\frac{1}{2}} + \rho_{(p_2+r_1)/(p_2+p_1)} A^{\frac{1}{2}}}{q[n]} A^{-\frac{1}{2}} \cdot \]

(iii) $\delta_n(p_n, r_n)$ is a decreasing function of both $r_n \geq 0$ and $p_n \geq \frac{\delta + r_1 + r_2 + ... + r_{n-1}}{q[n-1]}$, where $C_{A,B}[n]$ and $q[n]$ are defined as follows:

\[ C_{A,B}[n] = A^{k} \left\{ A^{\frac{p_1}{2}} \left[ A^{\frac{p_1}{2}} \left( A^{\frac{p_1}{2}} B^{p_1} A^{\frac{1}{2}} \right)^{p_2} A^{\frac{1}{2}} \right] \right\}^{p_n} A^{\frac{1}{2}} \]

and

\[ q[n] = \left[ \left( (p_1 + r_1) p_2 + r_2 \right) p_3 + ... + r_{n-1} \right] p_n + r_n . \]

We remark that (ii) can be considered as “a satellite inequality to chaotic order”.


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