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OPTIMAL ERROR ESTIMATES FOR CORRECTED TRAPEZOIDAL RULES

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Abstract. Corrected trapezoidal rules are proved for $\int_a^b f(x)\,dx$ under the assumption that $f'' \in L^p([a,b])$ for some $1 \leqslant p \leqslant \infty$. Such quadrature rules involve the trapezoidal rule modified by the addition of a term $k[f'(a) - f'(b)]$. The coefficient k in the quadrature formula is found that minimizes the error estimates. It is shown that when f' is merely assumed to be continuous then the optimal rule is the trapezoidal rule itself. In this case error estimates are in terms of the Alexiewicz norm. This includes the case when f'' is integrable in the Henstock–Kurzweil sense or as a distribution. All error estimates are shown to be sharp for the given assumptions on f''. It is shown how to make these formulas exact for all cubic polynomials f. Composite formulas are computed for uniform partitions.

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