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REFINEMENTS OF A TWO–SIDED INEQUALITY FOR TRIGONOMETRIC FUNCTIONS

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Abstract. In this paper, we prove that for $x \in (0, \pi/2)$

$$(\cos p_1 x)^{1/(3p_1^2)} < \frac{\sin x}{x} < (\cos p_0 x)^{1/(3p_0^2)} < \cdots < e^{-x^2/6} < \frac{2+\cos x}{3}$$

with the best constants $p_1 = 0.45346...$ and $p_0 = 1/\sqrt{5}$, and the function $p \mapsto (\cos px)^{1/(3p^2)}$ is decreasing on $(0,1)$. Our results greatly refine Adamović-Mitrinović's and Cusa's inequality. As applications, some precise estimations for certain special functions and constants are presented.

Mathematics subject classification (2010): Primary 26D05; Secondary 26A48, 26D15, 33B10.
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