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ESTIMATES FOR NEUMAN–SÁNDOR MEAN BY POWER MEANS AND THEIR RELATIVE ERRORS

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Abstract. For $a,b>0$ with $a\neq b$, let $NS(a,b)$ denote the Neuman-Sándor mean defined by

$$NS(a,b)=\frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}}$$

and $A_p(a,b)$, $\mathscr{L}_p(a,b)$ denote the r-order power and Lehmer means. Based on our earlier worker [27], we prove that

$$\alpha_p A_p<NS<A_p \text{ and } A_p<NS\leqslant\beta_p A_p$$

holds if and only if $p\geqslant 4/3$ and $p\leqslant p_0$, respectively, where

$$\alpha_p=\left(2^{1/p-1}\right)/\ln(1+\sqrt{2}) \text{ if } p\in[1/4/3,\infty),$$

$$\beta_p=\begin{cases} NS(1,x_0)/A_p(1,x_0) & \text{if } p\in(1,p_0], \\ 2^{1/p-1}/\ln\left(1+\sqrt{2}\right) & \text{if } p\in(0,1], \\ \infty & \text{if } p\in(-\infty,0] \end{cases}$$

are the best constants, here x_0 is the unique root of the equation

$$NS(1,x)=\frac{A(1,x)A_2(1,x)}{\mathscr{L}_{p_0-1}(1,x)}$$

on $(0,1)$, and $p\mapsto\alpha_p A_p$ is decreasing on $(0,\infty)$. Also, we have

$$\alpha_{4/3}A_{4/3}<A_{p_0}<NS<A_{4/3}<\alpha_{4/3}^{-1}A_{p_0}.$$

Our results clearly are generations of known ones.

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