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OPTIMAL BOUNDS FOR TOADER MEAN IN TERMS OF ARITHMETIC AND CONTRAHARMONIC MEANS

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Abstract. We find the greatest value α_1 and α_2, and the least values β_1 and β_2, such that the double inequalities $\alpha_1 C(a,b)+(1-\alpha_1)A(a,b)<T(a,b)<\beta_1 C(a,b)+(1-\beta_1)A(a,b)$ and $\alpha_2/A(a,b)+(1-\alpha_2)/C(a,b)<1/T(a,b)<\beta_2/A(a,b)+(1-\beta_2)/C(a,b)$ hold for all $a,b>0$ with $a\neq b$. As applications, we get new bounds for the complete elliptic integral of the second kind. Here, $C(a,b)=(a^2+b^2)/(a+b)$, $A(a,b)=(a+b)/2$, and

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$$

denote the contraharmonic, arithmetic, and Toader means of two positive numbers a and b, respectively.

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