Journal of
Mathematical
Inequalities
Volume 9, Number 1 (2015), 85–99
doi:10.7153/jmi-09-08

OPTIMAL INEQUALITIES FOR THE CONVEX COMBINATION OF ERROR FUNCTION

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Abstract. For $\lambda \in (0,1)$ and $x,y>0$ we obtain the best possible constants p and r, such that

$$\operatorname{erf}(M_p(x,y;\lambda)) \leqslant \lambda \operatorname{erf}(x) + (1-\lambda)\operatorname{erf}(y) \leqslant \operatorname{erf}(M_r(x,y;\lambda))$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and $M_p(x,y;\lambda) = (\lambda x^p + (1-\lambda)y^p)^{1/p} (p \neq 0)$, $M_0(x,y;\lambda) = x^\lambda y^{1-\lambda}$ are error function and weighted power mean, respectively. Furthermore, using these results, we generalized and complement an inequality due to Alzer.

Mathematics subject classification (2010): 33B20, 26D15.
Keywords and phrases: Error function, power mean, functional inequalities.

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