Journal of
Mathematical
Inequalities
Volume 9, Number 2 (2015), 331–343 doi:10.7153/jmi-09-27

# AN OPTIMAL INEQUALITIES CHAIN FOR BIVARIATE MEANS

ZHEN-HANG YANG AND YU-MING CHU

*Abstract.* Let $p \in \mathbb{R}$, $M$ be a bivariate mean, and $M_p$ be defined by $M_p(a,b) = M^{1/p}(a^p,b^p)$ $(p \neq 0)$ and $M_0(a,b) = \lim_{p\to 0} M_p(a,b)$. In this paper, we prove that the sharp inequalities $L_2(a,b) < P(a,b) < NS_{1/2}(a,b) < He(a,b) < A_{2/3}(a,b) < I(a,b) < Z_{1/3}(a,b) < Y_{1/2}(a,b)$ hold for all $a,b > 0$ with $a \neq b$, where $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $NS(a,b) = (a-b)/[2\operatorname{arcsinh}((a-b)/(a+b))]$, $He(a,b) = (a+\sqrt{ab}+b)/3$, $A(a,b) = (a+b)/2$, $I(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$, $Z(a,b) = a^{a/(a+b)}b^{b/(a+b)}$ and $Y(a,b) = I(a,b)e^{1-ab/L^2(a,b)}$ are respectively the logarithmic, first Seiffert, Neuman-Sándor, Heronian, arithmetic, identric, power-exponential and exponential-geometric means of $a$ and $b$.

*Mathematics subject classification* (2010): 26E60.
*Keywords and phrases*: Logarithmic mean, first Seiffert mean, Neuman-Sándor mean, identric mean, power-exponential mean, exponential-geometric mean.

## REFERENCES

[1] D. ACU, *Some inequalities for certain means in two arguments*, Gen. Math. **9**, 1-2 (2001), 11–14.
[2] H. ALZER AND W. JANOUS, *Solution of problem 8**, Crux Math. **13** (1987), 173–178.
[3] P. S. BULLEN, D. S. MITRINOVIĆ AND P. M. VASIĆ, *Means and Their Inequalities*, D. Reidel, Dordrecht, 1988.
[4] Y.-M. CHU, M.-K. WANG AND Y.-F. QIU, *An optimal double inequality between power-type Heron and Seiffert means*, J. Inequal. Appl. **2010**, Article ID 146945, 11 pages.
[5] Y.-M. CHU AND B.-Y. LONG, *Bounds of the Neuman-Sándor mean using power and identric means*, Abstr. Appl. Anal. **2013**, Article ID 832591, 6 pages.
[6] Y.-M. CHU, B.-Y. LONG, W.-M. GONG AND Y.-Q. SONG, *Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means*, J. Inequal. Appl. **2013**, 10 (2013), 13 pages.
[7] I. COSTIN AND G. TOADER, *Optimal evaluations of some Seiffert-type means by power means*, Appl. Math. Comput. **219**, 9 (2013), 4745–4754.
[8] I. COSTIN AND G. TOADER, *A nice separation of some Seiffert-type means by power means*, Int. J. Math. Math. Sci. **2012**, Article ID 430692, 6 pages.
[9] C. GINI, *Di una formula comprensiva delle media*, Metron **13**, 2 (1938), 3–22.
[10] P. A. HÄSTÖ, *A monotonicity property of ratios of symmetric homogeneous means*, JIPAM. J. Inequal. Pure Appl. Math. **3**, 5 (2002), Article 71, 23 pages.
[11] P. A. HÄSTÖ, *Optimal inequalities between Seiffert's mean and power mean*, Math. Inequal. Appl. **7**, 1 (2004), 47–53.
[12] A. A. JAGERS, *Solution of problem 887*, Nieuw Arch. Wisk. (4) **12** (1994), 230–231.
[13] G. JIA AND J.-D. CAO, *A new upper bound of the logarithmic mean*, JIPAM. J. Inequal. Pure Appl. Math. **4**, 4 (2003), Article 80, 4 pages.
[14] T. P. LIN, *The power mean and the logarithmic mean*, Amer. Math. Monthly **81** (1974), 879–883.
[15] E. NEUMAN AND J. SÁNDOR, *On the Schwab-Borchardt mean II*, Math. Pannon. **17**, 1 (2006), 49–59.
[16] Z. PÁLES, *Inequalities for sums of powers*, J. Math. Anal. Appl. **131**, 1 (1988), 265–270.
[17] A. O. PITTINGER, *Inequalities between arithmetic and logarithmic means*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **678-715** (1980), 15–18.

[18] K. B. STOLARSKY, *The power and generalized logarithmic means*, Amer. Math. Monthly **87**, 7 (1980), 545–548.

[19] J. SÁNDOR, *On certain inequalities for means III*, Arch. Math. **76**, 1 (2001), 34–40.

[20] J. SÁNDOR, *A note on the Gini means*, Gen. Math. **12**, 4 (2004), 17–21.

[21] H.-J. SEIFFERT, *Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen*, Elem. Math. **42** (1987), 105–107.

[22] H.-J. SEIFFERT, *Aufgabe $\beta$ 16*, Die Wurzel **29** (1995), 221–222.

[23] S. TOADER, *Derivatives of generalized means*, Math. Inequal. Appl. **5** (2002), 517–532.

[24] ZH.-H. YANG, *On the homogeneous functions with two parameters and its monotonicity*, JIPAM. J. Inequal. Pure Appl. Math. **6**, 4 (2005), Article 101, 11 pages.

[25] ZH.-H. YANG, *On the log-convexity of two-parameter homogeneous functions*, Math. Inequal. Appl. **10**, 3 (2007), 499–516.

[26] ZH.-H. YANG, *On the monotonicity and log-convexity of a four-parameter homogeneous mean*, J. Inequal. Appl. **2008**, Article ID 149286, 12 pages.

[27] ZH.-H. YANG, *Sharp bounds for the second Seiffert mean in terms of power means*, arXiv: 1206.5494v1 [math. CA].

[28] ZH.-H. YANG, *Sharp power means bounds for Neuman-Sándor mean*, arXiv: 1208.0895v1 [math. CA].

[29] ZH.-H. YANG, *Estimates for Neuman-Sándor mean by power means and their relative errors*, J. Math. Inequal. **7**, 4 (2013), 711–726.



Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com