ASYMPTOTIC EXPANSIONS OF GAMMA AND RELATED FUNCTIONS, BINOMIAL COEFFICIENTS, INEQUALITIES AND MEANS

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Abstract. We give an overview of the use of asymptotic expansions of gamma and related functions — ratio of gamma functions, powers, digamma and polygamma functions. The aim is to introduce a general theory which can unify various particular formulas for factorial functions and binomial coefficients. The connection with inequalities for gamma function is established. Also, a systematic approach to asymptotic expansion of various integral means, bivariate classical and parameter means is given, with applications to comparison of means.

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