

A CHARACTERIZATION OF MATRIX INEQUALITY $A \geq B \geq C$ VIA KARCHER MEAN

JIAN SHI AND SIYUAN SHEN

Abstract. Let A, B and C be three positive definite matrices. In this paper, we show a characterization of $A \geq B \geq C$ via the Karcher mean as follows,

$$\Lambda(\omega; A^{-p_1}, B^{-p_2}, B^t \sharp_s C^{p_3}) \leq C,$$

$$\Lambda(\omega; C^{-p_1}, B^{-p_2}, B^t \sharp_s A^{p_3}) \geq A$$

hold for $t \in [0, 1]$, $s \geq 1$, $p_1, p_2 > 0$ and $p_3 > 1$, where $\hat{\omega} = (\frac{1}{p_1+1}, \frac{1}{p_2+1}, \frac{2}{(p_3-t)s+t-1})$, $\omega = (w_1, w_2, w_3) = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

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REFERENCES

- [1] T. ANDO, C. K. LI AND R. MATHIAS, *Geometric means*, Linear Algebra Appl., **385** (2004), 305–334.
- [2] R. BHATIA AND J. HOLBROOK, *Riemannian geometry and matrix geometric means*, Linear Algebra Appl., **413** (2006), 594–618.
- [3] R. BHATIA, *Positive definite matrices*, Princeton Ser. Appl. Math., Princeton University Press, Princeton, NJ, 2007.
- [4] M. ITO, *Matrix inequality including grand Furuta inequality via Karcher mean*, J. Math. Inequal., **2** (2014), 279–285.
- [5] J. D. LAWSON AND Y. LIM, *Monotonic properties of the least squares mean*, Math. Ann., **351** (2011), 267–279.
- [6] J. D. LAWSON AND Y. LIM, *Karcher means and Karcher equations of positive definite operators*, Trans. Amer. Math. Soc., Series B, **1** (2014), 1–22.
- [7] Y. LIM AND M. PÁLFIA, *Matrix power means and Karcher mean*, J. Funct. Anal., **262** (2012), 1498–1514.
- [8] C.-S. LIN AND Y. J. CHO, *Characterizations of operator inequality $A \geq B \geq C$* , Math. Inequal. Appl., **14** (2011), 575–580.
- [9] M. MOAKHER, *A differential geometric approach to the geometric mean of symmetric positive-definite matrices*, SIAM J. Matrix Anal. Appl., **26** (2005), 735–747.
- [10] M. UCHIYAMA, *Criteria for operator means*, J. Math. Soc. Japan, **55** (2003), 197–207.
- [11] T. YAMAZAKI, *The Riemannian mean and matrix inequalities related to the Ando-Hiai inequality and chaotic order*, Oper. Matrices, **3** (2012), 577–588.
- [12] T. YAMAZAKI, *An elementary proof of arithmetic-geometric mean inequality of the weighted Riemannian mean of positive definite matrices*, Linear Algebra Appl., **438** (2013), 1564–1569.
- [13] J. YUAN AND C. WANG, *Riccati type operator equation and Furuta's question*, Math. Inequal. Appl., **4** (2014), 1337–1342.