

## ON THE ROLE OF ULTRAMODULARITY AND SCHUR CONCAVITY IN THE CONSTRUCTION OF BINARY COPULAS

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*Abstract.* We discuss and stress the role of ultramodularity and Schur concavity in special types of constructions of copulas. After recalling some known ultramodularity-based results, we focus on the so-called  $D$ -product of a copula and its dual. We show that for each copula  $D$  which is ultramodular and Schur concave on the left upper triangle of the unit square, this  $D$ -product of an arbitrary copula and its dual is again a copula. Several examples and counterexamples are given. Finally, some of our results are generalized to the case of semicopulas and quasi-copulas.

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