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MODULARITY BOUNDS FOR CLUSTERS LOCATED BY LEADING EIGENVECTORS OF THE NORMALIZED MODULARITY MATRIX

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Abstract. Nodal theorems for generalized modularity matrices ensure that the cluster located by the positive entries of the leading eigenvector of various modularity matrices induces a connected subgraph. In this paper we obtain lower bounds for the modularity of that subgraph showing that, under certain conditions, the nodal domains induced by eigenvectors corresponding to highly positive eigenvalues of the normalized modularity matrix have indeed positive modularity, that is, they can be recognized as modules inside the network. Moreover we establish Cheeger-type inequalities for the cut-modularity of the graph, providing a theoretical support to the common understanding that highly positive eigenvalues of modularity matrices are related with the possibility of subdividing a network into communities.

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