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OPTIMAL INEQUALITIES INVOLVING POWER–EXPONENTIAL MEAN, ARITHMETIC MEAN AND GEOMETRIC MEAN

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Abstract. For $a,b>0$ with $a\neq b$, the power-exponential mean is defined by

$$Z\equiv Z(a,b)=\exp\left(\frac{a\ln a+b\ln b}{a+b}\right)=\sqrt{ab}e^{t\tanh t},$$

where $t=\ln\sqrt{a/b}$. In this paper, we prove the double inequality

$$\left(\frac{Z^p+G^p}{2}\right)^{1/p}<A<\left(\frac{Z^q+G^q}{2}\right)^{1/q}$$

holds for $a,b>0$, $a\neq b$ with the best constants $p=2/3$ and $q=1$, where $A=(a+b)/2$, $G=\sqrt{ab}$. We also establish the sharp bounds for $e^{t\tanh t}$ as follows:

$$1<\frac{e^{t\tanh t}}{2\cosh t-1}<1.055,$$
$$\frac{1}{\sqrt{2}}<\frac{e^{t\tanh t}}{2(\cosh t)^{2/3}-1}<1$$

for $t>0$. These improve some known results.

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