MONOTONICITY AND SHARP INEQUALITIES RELATED TO GAMMA FUNCTION

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Abstract. In this paper, we investigate the monotonicity pattern of the function

$$x \mapsto \frac{\ln \Gamma(x + 1)}{\ln(x^2 + a) - \ln(x + a)}$$

on $$(0, 1)$$ for $$a \geq 1$$ and resolve an open problem. From which we prove that the double inequality

$$\left(\frac{x^2 + a}{x + a}\right)^{(1 - \gamma)(a + 1)} < \Gamma(x + 1) < \left(\frac{x^2 + b}{x + b}\right)^{(1 - \gamma)(b + 1)}$$

holds for $$x \in (0, 1)$$ if and only if $$0 < a \leq (1 - \gamma)/(2\gamma - 1)$$ and $$b \geq (\pi^2 - 6\gamma)/(18 - 12\gamma - \pi^2)$$, where

while the double inequality

$$\left(\frac{x^2 + a}{x + a}\right)^{\gamma a} < \Gamma(x + 1) < \left(\frac{x^2 + b}{x + b}\right)^{\gamma b}$$

holds for $$x \in (0, 1)$$ if and only if $$a \geq (1 - \gamma)/(2\gamma - 1)$$ and $$0 < b \leq 6\gamma/(\pi^2 - 12\gamma)$$, where $$\gamma = 0.577\ldots$$ denotes Euler-Mascheroni’s constant. These greatly improve some existing results.


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REFERENCES