

MONOTONICITY AND SHARP INEQUALITIES RELATED TO GAMMA FUNCTION

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Abstract. In this paper, we investigate the monotonicity pattern of the function

$$x \mapsto \frac{\ln \Gamma(x+1)}{\ln(x^2+a) - \ln(x+a)}$$

on $(0, 1)$ for $a \geq 1$ and resolve an open problem. From which we prove that the double inequality

$$\left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{(1-\gamma)(b+1)}$$

holds for $x \in (0, 1)$ if and only if $0 < a \leq (1-\gamma)/(2\gamma-1)$ and $b \geq (\pi^2 - 6\gamma)/(18 - 12\gamma - \pi^2)$, while the double inequality

$$\left(\frac{x^2+a}{x+a}\right)^{\gamma a} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{\gamma b}$$

holds for $x \in (0, 1)$ if and only if $a \geq (1-\gamma)/(2\gamma-1)$ and $0 < b \leq 6\gamma/(\pi^2 - 12\gamma)$, where $\gamma = 0.577\dots$ denotes Euler-Mascheroni's constant. These greatly improve some existing results.

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