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A FAMILY OF WINDSCHITL TYPE APPROXIMATIONS FOR GAMMA FUNCTION

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Abstract. In this paper, we present a family of high accurate approximation formulas

$$\mathscr{W}_p(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}\right)$$

for gamma function $\Gamma(x+1)$ with parameter $p \geqslant -33/35$, and prove the function

$$x \mapsto \ln\Gamma(x+1) - \ln\mathscr{W}_p(x)$$

is strictly increasing and concave on $(0,\infty)$ if and only if $p \geqslant 158/315$. This yields some new sharp approximations for gamma function.

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