

## MONOTONICITY AND INEQUALITIES INVOLVING THE INCOMPLETE GAMMA FUNCTION

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*Abstract.* In this paper, by observing the monotonicity of three ratios involving the integral function  $\int_0^x e^{-t^p} dt$  for  $p, x > 0$ , we offer some new sharp bounds for the incomplete gamma function, which greatly improve and extend some known results. Also, as by-products, we unexpectedly obtain two power series representations for the incomplete gamma function.

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