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UPPER AND LOWER BOUNDS FOR THE OPTIMAL CONSTANT IN THE EXTENDED SOBOLEV INEQUALITY. DERIVATION AND NUMERICAL RESULTS

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Abstract. We prove and give numerical results for two lower bounds and eleven upper bounds to the optimal constant $k_0 = k_0(n,\alpha)$ in the inequality

$$\|u\|_{2n/(n-2\alpha)} \leqslant k_0 \, \|\nabla u\|_2^{\alpha} \, \|u\|_2^{1-\alpha}, \qquad u \in H^1(\mathbb{R}^n),$$

for $n = 1$, $0 < \alpha \leqslant 1/2$, and $n \geqslant 2$, $0 < \alpha < 1$.

This constant k_0 is the reciprocal of the infimum $\lambda_{n,\alpha}$ for $u \in H^1(\mathbb{R}^n)$ of the functional

$$\Lambda_{n,\alpha} = \frac{\|\nabla u\|_2^{\alpha} \, \|u\|_2^{1-\alpha}}{\|u\|_{2n/(n-2\alpha)}}, \qquad u \in H^1(\mathbb{R}^n),$$

where for $n = 1$, $0 < \alpha \leqslant 1/2$, and for $n \geqslant 2$, $0 < \alpha < 1$.

The lowest point in the point spectrum of the Schrödinger operator $\tau = -\Delta + q$ on $\mathbb{R}^n$ with the real-valued potential q can be expressed in $\lambda_{n,\alpha}$ for all $q_- = \max(0,-q) \in L^p(\mathbb{R}^n)$, for $n = 1$, $1 \leqslant p < \infty$, and $n \geqslant 2$, $n/2 < p < \infty$, and the norm $\|q_-\|_p$.

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