Journal of Mathematical Inequalities
Volume 13, Number 3 (2019), 833–854
doi:10.7153/jmi-2019-13-58

# A SURVEY FOR GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

LI YIN, LI-GUO HUANG, YONG-LI WANG AND XIU-LI LIN

*Abstract.* The generalized trigonometric functions which have a short history, were introduced by Lindqvist two decades ago. Since 2012, many mathematician began to study their classical inequalities, general convexity and concavity, multiple-angle formulas and parameter convexity and concavity. A number of results have been obtained. This is a survey. Some new refinements, generalizations, applications, and related problems are summarized.

*Mathematics subject classification* (2010): 33B10, 33C05, 33C99, 33B99, 33E30, 33C20, 33E05, 34L10, 34L40, 26D15.

*Keywords and phrases*: Generalized trigonometric and hyperbolic functions, inequalities, generalized elliptic integrals, hypergeometric function, generalized convexity and concavity.

## REFERENCES

[1] M. ABRAMOWITZ, I. STEGUN, EDS., *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards, Dover, New York, 1965.

[2] H. ALZER, K. RICHARDS, *A note on a function involving complete elliptic integrals: Monotonicity, convexity, inequalities*, Anal. Math., **41**(2015), 133–139.

[3] G. E. ANDREWS, R. ASKEY AND R. ROY, *Special functions*, Cambridge University Press, Cambridge, 1999.

[4] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Genenalized convexity and inequalities*, J. Math. Anal. Appl., **335** (2007), 1294–1308.

[5] Á. BARICZ, B. A. BHAYO, T. K. POGÁNY, *Functional inequalities for generalized inverse trigonometric and hyperbolic functions*, J. Math. Anal. Appl., **417** (2014), 244–259, http://arxiv.org/abs/1401.4863.

[6] Á. BARICZ, B. A. BHAYO, M. VUORINEN, *Turán type inequalities for generalized inverse trigonometric functions*, Filomat, **29**, No. 2 (2015), 303–313, http://arxiv.org/abs/1209.1696.

[7] Á. BARICZ, B. A. BHAYO, R. KLÉN, *Convexity properties of generalized trigonometric and hyperbolic functions*, Aequat. Math., **89** (2015), 473–484, http://arxiv.org/abs/1301.0699.

[8] F. D. BURGOYNE, *Generalized trigonometric functions*, Math. Comp., **18** (1964), 314–316.

[9] Á. BARICZ, *Turán type inequalities for generalized complete elliptic integrals*, Math. Z., **256** (2007), 895–911.

[10] Á. BARICZ, *Geometrically concave univariate distributions*, J. Math. Anal. Appl., **363**, No. 1 (2010), 182–196.

[11] P. J. BUSHELL, D. E. EDMUNDS, *Remarks on generalised trigonometric functions*, Rocky Mountain J. Math., **42** (2012), 13–52.

[12] B. A. BHAYO, J. SÁNDOR, *Inequalities connecting generalized trigonometric functions with their inverses*, Issues of Analysis, **2**, No. 20 (2013), 82–90.

[13] B. A. BHAYO, M. VUORINEN, *On generalized trigonometric functions with two parameters*, J. Approx. Theory, **164** (2012), 1415–1426, http://arxiv.org/abs/1112.0483.

[14] B. A. BHAYO, M. VUORINEN, *Inequalities for eigenfunctions of the $p$-Laplacian*, Issues of Analysis, **2**, No. 20 (2013), 13–35, http://arxiv.org/abs/1101.3911.

[15] B. A. BHAYO, M. VUORINEN, *Power mean inequalities generalized trigonometric functions*, Math. Vesnik, **67**, No. 1 (2015), 17–25, http://arxiv.org/abs/1209.0983.


Paper JMI-13-58

[16] B. A. BHAYO, M. VUORINEN, *On generalized complete elliptic integrals and modular functions*, Proc. Edinb. Math. Soc., **55**(2012), 591–611, http://arxiv.org/abs/1102.1078.
[17] B. A. BHAYO AND L. YIN, *Logarithmic mean inequality for generalized trigonometric and hyperbolic functions*, Acta. Univ. Sapientiae Math., **6**, No. 2 (2014), 135–145, http://arxiv.org/abs/1404.6732.
[18] B. A. BHAYO AND L. YIN, *On the generalized convexity and concavity*, Problemy Analiza-Issues of Analysis, **22**, No. 1 (2015), 1–9, http://arxiv.org/abs/1411.6586.
[19] B. A. BHAYO AND L. YIN, *On the conjecture of generalized trigonometric and hyperbolic functions*, Math. Pannon., Vol. 24, No. 2 (2013), 1–8, http://arxiv.org/abs/1402.7331.
[20] B. A. BHAYO AND L. YIN, *On generalized* $(p,q)$ *elliptic integrals*, http://arxiv.org/abs/1507.00031
[21] B. A. BHAYO AND L. YIN, *On a function involving generalized complete* $(p,q)$ *elliptic integrals*, http://arxiv.org/abs/1606.03621.
[22] W. Y. CUI AND L. YIN, *Logarithmic mean inequalities for the generalized trigonometric and hyperbolic functions with two parameters*, Octogon Math. Mag., **22**, No. 2 (2014), 700–705.
[23] P. DRÁBEK, R. MANÁSEVICH, *On the closed solution to some* $p-$*Laplacian nonhomogeneous eigenvalue problems*, Diff. and Int. Eqns., **12** (1999), 723–740.
[24] D. E. EDMUNDS AND J. LANG, *Generalized trigonometric functions from different points of view*, Progresses in Mathematics, Physics and Astronomy(Pokroky MFA), **4** (2009).
[25] D. E. EDMUNDS, P. GURKA, J. LANG, *Properties of generalized trigonometric functions*, J. Approx. Theory, **164** (2012), 47–56.
[26] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, F. G. TRICOMI, *Higher Transcendental Functions*, vol. I, Melbourne, 1981.
[27] V. HEIKKALA, H. LINDÉN, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integrals and the Legendre M-function*, J. Math. Anal. Appl., **338** (2008), 223–243, arXiv:math/0701438.
[28] V. HEIKKALA, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integrals*, Comput. Methods Funct. Theory, **9**, No. 1(2009), 75–109, arXiv:math/0701436.
[29] W. D. JIANG, M. K. WANG, Y. M. CHU, Y. P. JIANG, F. QI, *Convexity of the generalized sine function and the generalized hyperbolic sine function*, J. Approx. Theory, **174** (2013), 1–9.
[30] J. C. KUANG, *Applied inequalities (Second edition)*, Shan Dong Science and Technology Press, Jinan, 2002.
[31] D. B. KARP AND E. G. PRILEPKINA, *Parameter convexity and concavity of generalized trigonometric functions*, J. Math. Anal. Appl., **421**, No. 1 (2015), 370–382, http://arxiv.org/abs/1402.3357.
[32] R. KLÉN, M. VUORINEN, X.-H. ZHANG, *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl., **409** (2014), 521–529, http://arxiv.org/abs/1210.6749.
[33] J. LANG AND D. E. EDMUNDS, *Eigenvlues, Embeddings and generalized trigonometric functions*, Lecture Notes in Mathematics, 2016, Springer, 2011.
[34] P. LINDQVIST, *Some remarkable sine and cosine functions*, Ricerche di Math., **XLIV** (1995), 269–290.
[35] P. LINDQVIST, J. PEETRE, *p-arclength of the q-circle*, Math. Stud., **72** (2003), 139–145.
[36] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
[37] E. NEUMANN, *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Inequal., **8**, no. 4 (2014), 725–736.
[38] E. NEUMAN, *Inequalities and bounds for generalized complete elliptic integrals*, J. Math. Anal. Appl., **373** (2011), 203–213.
[39] E. NEUMANN, *On the inequalities for the generalized trigonometric functions*, Int. J. Anal., **2014**, (2014), 1–5.
[40] E. NEUMANN AND J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*, Math. Inequal. Appl., **13**, No. 4 (2010), 715–723.
[41] F. QI, D. W. NIU AND B. N. GUO, *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl., **2009** (2009), Article ID 271923, 1–52.
[42] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, C. W. CLARK, EDS., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
[43] D. SHELUPSKY, *A generalization of trigonometric functions*, Amer. Monthly, **66**, No. 10 (1959), 879–884.

[44] L. J. SLATER, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.
[45] Y. Q. SONG, Y. M. CHU, B. Y. LIU AND M. K. WANG, *A note on generalized trigonometric and hyperbolic functions*, J. Math. Inequal., **8**, No. 3 (2014), 635–642.
[46] G. WANG, X. ZHANG, AND Y. CHU, *Inequalities for the generalized elliptic integrals and modular functions*, J. Math. Anal. Appl., **331**, No. 2 (2007), 1275–1283.
[47] S. TAKEUCHI, *Generalized Jacobian elliptic functions and their application to bifurcation problems associated with $p$-Laplacian*, J. Math. Anal. Appl., **385** (2012), 24–35.
[48] T. KAMIYA AND S. TAKEUCHI, *Complete $(p,q)$-elliptic integrals with application to a family of means*, J. Classical Anal.,, **10**, No. 1:15–25, http://arxiv.org/abs/1507.01383.
[49] S. TAKEUCHI, *The complete $p$-elliptic integrals and a computation formula of $\pi_p$ for $p=4$*, Ramanujan Journal, 2018, 46 (2) :309–321, http://arxiv.org/abs/1503.02394.
[50] S. TAKEUCHI, *A new form of the generalized complete elliptic integrals*, Kodai J. Math., **39**, No. 1 (2016), 202–226, http://arxiv.org/abs/1411.4778.
[51] S. TAKEUCHI, *Multiple-angle formulas of generalized trigonometric functions with two parameters*, http://arxiv.org/abs/1603.06709.
[52] S. TAKEUCHI, *Legendre-type relations for generalized complete elliptic integrals*, Journal of Classical Analysis, **9**, No. 1 (2016), 35–42, http://arxiv.org/abs/1606.05115.
[53] C. Y. YANG, *Inequalities for generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl., **419** (2014), 775–782.
[54] L. YIN AND L. G. HUANG, *Some inequalities for the generalized sine and generalized hyperbolic sine*, J. Classical Anal., **3**, No. 1 (2013), 85–90.
[55] L. YIN AND L. G. HUANG, *A new inequalities and several conjectures for the generalized functions*, Octogon Math. Mag., **21**, No. 2 (2013), 564–568.
[56] L. YIN AND L. G. HUANG, *Inequalities for generalized trigonometric and hyperbolic functions with two parameters*, J. Nonlinear Sci. Appl., **8**, No. 4 (2015), 315–323.
[57] L. YIN AND L. G. HUANG, *Some New Wilker and Cusa Type Inequalities for Generalized Trigonometric and Hyperbolic Functions*, J. Inequal. Appl., **2018**, 2018: 52, http://rgmia.org/papers/v18/v18a29.pdf.
[58] L. YIN AND L. G. HUANG, *Inequalities for generalized trigonometric and hyperbolic functions with two parameters*, Pure Appl. Math., **31**, No. 5 (2015), 474–479.
[59] L. YIN, L. G. HUANG AND F. QI, *Some inequalities for the generalized trigonometric and generalized hyperbolic functions*, Turnish J. Anal. number theory, **2**, No. 3 (2014), 96–101.
[60] L. YIN AND L. F. MI, *Landen type inequalities for generalized complete elliptic integrals*, Adv. Stud. Comtem. Math., **26**, No. 4 (2012), 717–722.



Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com