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INEQUALITIES FOR GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS WITH ONE PARAMETER

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Abstract. In the article, we establish several new inequalities for the generalized trigonometric and hyperbolic functions with one parameter, generalize the well known Mitrinović-Adamović, Lazarević, Huygens-type, Wilker-type and Cusa-Huygens-type inequalities to the cases of the generalized trigonometric and hyperbolic functions with one parameter.

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