

## UPPER BOUND OF THE THIRD HANKEL DETERMINANT FOR A SUBCLASS OF $q$ -STARLIKE FUNCTIONS ASSOCIATED WITH THE LEMNISCATE OF BERNOULLI

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*Abstract.* In this paper, we use  $q$ -derivative operator to define a new subclass of starlike functions related with the lemniscate of Bernoulli. For this function class we obtain upper bound of the third Hankel determinant. For validity of our results, relevant connections with those in earlier works are also pointed out.

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