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THE DIRICHLET PROBLEM FOR A SUB–ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY ON THE HEISENBERG GROUP

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Abstract. This paper studies the following singular sub-elliptic equation:

$$\begin{cases} -\Delta_H u = \frac{h(\xi)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{H}^n$ is a smooth bounded domain, $\gamma > 0$ and $h \geqslant 0$. We first use the Schauder's fixed point theorem and approximating method to prove the existence of solutions to the above equation. We then obtain the uniqueness result by proving a weak comparison principle and further deduce that the solution is cylindrically symmetric under some necessary structural conditions on Ω and h.

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