HARDY INEQUALITIES AND CAFFARELLI–KOHN–NIRENBERG INEQUALITIES WITH RADIAL DERIVATIVE

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Abstract. In this paper, we study several inequalities of Hardy and Caffarelli-Kohn-Nirenberg type. We set up some optimal versions of these inequalities using the radial derivatives or the convex combinations of the full gradient and its radial part. We also exhibit their optimizers in some certain cases.

Mathematics subject classification (2010): 26D10, 35A23, 46E35.

Keywords and phrases: Hardy inequalities, Caffarelli-Kohn-Nirenberg inequalities, best constants, optimizers, radial derivative.

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