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A TRUDINGER–MOSER TYPE INEQUALITY AND ITS EXTREMAL FUNCTIONS IN DIMENSION TWO

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Abstract. Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $W_0^{1,2}(\Omega)$ be the usual Sobolev space and $\lambda(\Omega)$ be the first eigenvalue of the Laplace-Beltrami operator, say

$$\lambda(\Omega) = \inf_{u \in W_0^{1,2}(\Omega),\, \int_\Omega u^2 dx = 1} \int_\Omega |\nabla u|^2 dx.$$

Using blow-up analysis, we prove that for real numbers $\alpha < \lambda(\Omega)$ and $\beta < 4\pi$, the supremum

$$\sup_{u \in W_0^{1,2}(\Omega),\, \int_\Omega |\nabla u|^2 dx - \alpha \int_\Omega u^2 dx \leqslant 1} \int_\Omega (e^{4\pi u^2} - \beta u^2) dx$$

can be attained by some function $u \in W_0^{1,2}(\Omega)$ with $\int_\Omega |\nabla u|^2 dx - \alpha \int_\Omega u^2 dx = 1$. In the case $\beta = 0$, this is reduced to a result of Yang [24].

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