Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1083–1094
doi:10.7153/jmi-2020-14-70

SOME NECESSARY AND SUFFICIENT CONDITIONS FOR A VMO FUNCTION

GUANGQING WANG AND JINHUI LI*

Abstract. Let $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ $(W\widetilde{L}_{q,\lambda}(\mathbb{R}^n))$ be the (weak) modified Morrey spaces. In this paper, for some appropriate indices p, λ and q, we firstly prove that the commutator $[b, I_\alpha]$, generated by the symbol b and the fractional intergal operator I_α, is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if b belongs to $\mathrm{VMO}(\mathbb{R}^n)$. Besides, for the fractional maximal commutator $M_{\alpha,b}$, the result still holds. Moreover, commutators of fractional maximal functions with symbol b are investigated. More precisely, it is shown that commutators $[b, M_\alpha]$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if b belongs to $\mathrm{VMO}(\mathbb{R}^n)$ with the negative part of b equals to zero almost everywhere.

Mathematics subject classification (2010): 42B35, 47B47.
Keywords and phrases: Characterization, commutator, VMO.

REFERENCES

[1] F. JOHN, L. NIRENBERG, *On functions of bounded mean oscillation*, Communications on pure and applied Mathematics, 1961, 14 (3): 415–426.
[2] D. SARASON, *Functions of vanishing mean oscillation*, Transactions of the American Mathematical Society, 1975, 207: 391–405.
[3] S. CHANILLO, *A note on commutators*, Indiana University Mathematics Journal, 1982, 31 (1): 7–16.
[4] Y. DING, *A characterization of BMO via commutators for some operators*, Northeast. Math. J, 1997, 13 (4): 422–432.
[5] S. SHIRAI, *Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces*, Hokkaido mathematical journal, 2006, 35 (3): 683–696.
[6] G. DI FAZIO, M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital, 1991, 5 (7): 323–332.
[7] Y. KOMORI, T. MIZUHARA, *Notes on commutators and Morrey spaces*, Hokkaido mathematical journal, 2003, 32 (2): 345–353.
[8] D. R. ADAMS, *Commutators of Morrey Potentials Morrey Spaces*, Springer International Publishing, 2015: 77–83.
[9] R. R. COIFMAN, R. ROCHBERG, G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Annals of Mathematics, 1976, 103: 611–635.
[10] S. JANSON, M. TAIBLESON, G. WEISS, *Elementary characterizations of the Morrey-Campanato spaces*, Lecture Notes in Math, 1983, 992: 101–114.
[11] M. PALUSZYŃSKI, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana University Mathematics Journal, 1995, 44: 1–17.
[12] S. SHI, S. LU, *Some characterizations of Campanato spaces via commutators on Morrey spaces*, Pacific Journal of Mathematics, 2013, 264 (1): 221–234.
[13] V.-S. GULIYEV AND K.-R. RAHIMOVA, *Parabolic fractional maximal operator in modified parabolic Morrey spaces*, J. Funct. Space. Appl, **2012** (2012).
[14] A. M. ALPHONSE, *An end point estimate for maximal commutators*, Journal of Fourier Analysis and Applications, 2000, 6 (4): 449–456.

Paper JMI-14-70

[15] J. GARCIA-CUERVA, E. HARBOURE, C. SEGOVIA, J. L. TORREA, *Weighted norm inequalities for commutators of strong singular integral*, Indiana University Mathematics Journal, 1991, 40 (4): 1397–1420.
[16] D. LI, G. HU, X. SHI, *Weighted norm inequalities for the maximal commutators of singular integral operators*, Journal of mathematical analysis and applications, 2006, 319 (2): 509–521.
[17] C. SEGOVIA, J. L. TORREA, *Weighted inequalities for commutators of fractional and singular integrals*, Publicacions Matemàtiques, 1991, 35 (1): 209–235.
[18] C. SEGOVIA, J. L. TORREA, *Higher order commutators for vector-valued Calderón-Zygmund operators*, Transactions of the American Mathematical Society, 1993, 336 (2): 537–556.
[19] M. MILMAN, T. SCHONBEK, *Second order estimates in interpolation theory and applications*, Proceedings of the American Mathematical Society, 1990, 110 (4): 961–969.
[20] J. BASTERO, M. MILMAN, F. RUIZ, *Commutators for the maximal and sharp functions*, Proceedings of the American Mathematical Society, 2000, 128 (11): 3329–3334.
[21] V. S. GULIYEV, J. J. HASANOV, Y. ZEREN, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, Journal of the Mathematical Inequalities, 2011, 5 (4): 491–506.
[22] S. JANSON, *Mean oscillation and commutators of singular integral operators*, Arkiv för Matematik, 1978, 16 (1): 263–270.
[23] R. R. COIFMAN, R. ROCHBERG, *Another characterization of BMO*, Proceedings of the American Mathematical Society, 1980: 249–254.
[24] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA, et al. *A note on maximal commutators and commutators of maximal functions*, Journal of the Mathematical Society of Japan, 2015, 67 (2): 581–593.
[25] C. XIE, *Some estimates of commutators*, Real Analysis Exchange, 2010, 36 (2): 405–416.
[26] P. ZHANG, J. WU, *Commutators of the fractional maximal function on variable exponent Lebesgue spaces*, Czechoslovak Mathematical Journal, 2014, 64 (1): 183–197.
[27] P. ZHANG, J. L. WU, *Commutators of the fractional maximal functions*, Acta Math. Sin., Chin. Ser, 2009, 52: 1235–1238.
[28] C. AYKOL, H. ARMUTCU, M. N. OMAROVA, *Maximal commutator and commutator of maximal function on modified Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics, 2016, 36 (1): 29–35.