SHARP POWER MEAN BOUNDS FOR THE
TANGENT AND HYPERBOLIC SINE MEANS

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Abstract. In the article, we prove that the double inequalities

$$M_{\alpha_1}(a,b) < M_{\tan}(a,b) < M_{\beta_1}(a,b),$$
$$M_{\alpha_2}(a,b) < M_{\sinh}(a,b) < M_{\beta_2}(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leqslant 1/3$, $\beta_1 \geqslant \log 2/\log(2\tan 1) \approx 0.61007$, $\alpha_2 \leqslant 2/3$ and $\beta_2 \geqslant \log 2/\log(2\sinh 1) \approx 0.81109$, where M_p, $M_{\tan}$ and $M_{\sinh}$ are the pth power mean, tangent mean and hyperbolic sine mean, respectively.

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