

## HYERS–ULAM STABILITY FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS OF CARATHÉODORY TYPE

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*Abstract.* This study deals with the Hyers–Ulam stability (HUS) of the second order linear differential equations  $x'' + \alpha x' + \beta x = f(t)$  without the assumption of continuity of  $f(t)$ . In particular, the main purpose of this study is to find a specific exact solution near the approximate solution, and the best HUS constant. Furthermore, the instability is also discussed, and a necessary and sufficient condition is obtained. Finally, a specific application example and a numerical simulation are presented.

*Mathematics subject classification (2020):* 34A12, 34A30, 34D10, 34H05.

*Keywords and phrases:* Hyers–Ulam stability, linear differential equation, Carathéodory type, control system, best HUS constant.

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