A FAMILY OF WILKER'S INEQUALITIES IN
TWO PARAMETERS AND ITS APPLICATIONS

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Abstract. Based on the reparameterization technique, this paper presents a family of Wilker's Inequalities with an extra parameter α. By using Pade approximation, the optimal value of α within a local interval is given. And simple proofs are provided based on the bounds of the functions $\sin(\alpha x)$ and $\cos(\alpha x)$. With the constraint $\alpha = 1$, the new method can recover several previous results. In principle, the idea can be extended to more other forms of bounding functions, or even more other inequality types. The comparison results show that the results in this paper are much better than those of prevailing methods.

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