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BOUNDING THE SÁNDOR–YANG MEANS FOR THE COMBINATIONS OF CONTRAHARMONIC AND ARITHMETIC MEANS

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Abstract. In the article, we prove that $t_1 = 1/2+\sqrt{2^{1/(2p)}e^{(\pi-4)/(4p)}-1}/2$, $t_2 = 1/2+\sqrt{6p}/(12p)$, $t_3 = 1/2+\sqrt{(1+\sqrt{2})^{\sqrt{2}/p}/e^{1/p}-1}/2$ and $t_4 = 1/2+\sqrt{3p}/(6p)$ are the best possible parameters on the interval $[1/2,1]$ such that the double inequalities

$$C^p[t_1u+(1-t_1)v,t_1v+(1-t_1)u]A^{1-p}(u,v) < Q(u,v)e^{\frac{A(u,v)}{\mathscr{T}(u,v)}-1}$$
$$< C^p[t_2u+(1-t_2)v,t_2v+(1-t_2)u]A^{1-p}(u,v),$$
$$C^p[t_3u+(1-t_3)v,t_3v+(1-t_3)u]A^{1-p}(u,v) < A(u,v)e^{\frac{Q(u,v)}{\mathscr{NS}(u,v)}-1}$$
$$< C^p[t_4u+(1-t_4)v,t_4v+(1-t_4)u]A^{1-p}(u,v)$$

hold for all $u,v>0$ with $u\neq v$ and $p\in[1/2,\infty)$, where $A(u,v)=(u+v)/2$, $Q(u,v)=\sqrt{(u^2+v^2)/2}$, $C(u,v)=(u^2+v^2)/(u+v)$, $\mathscr{T}(u,v)=(u-v)/[2\arctan((u-v)/(u+v))]$ and $\mathscr{NS}(u,v)=(u-v)/[2\sinh^{-1}((u-v)/(u+v))]$ are respectively the arithmetic, quadratic, contraharmonic, Seiffert and Neuman-Sándor means of u and v, and $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

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