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ABSOLUTELY MONOTONIC FUNCTIONS INVOLVING THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND WITH APPLICATIONS

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Abstract. Let $\mathscr{K}(r)$ be the complete elliptic integral of the first kind. In this paper, we prove that the function $F_p(x) = (1-x)^p \exp \mathscr{K}(\sqrt{x})$ is absolutely monotonic on $(0,1)$ if and only if $p \leqslant \pi/8$, and $-F_p'(x)$ is absolutely monotonic on $(0,1)$ if and only if $1/2 \leqslant p \leqslant \left(\pi+4+\sqrt{16-\pi}\right)/8$. This generalizes a known result and gives several new inequalities involving the complete elliptic integral of the first kind.

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