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A HILBERT–TYPE INEQUALITY IN THE WHOLE PLANE WITH THE CONSTANT FACTOR RELATED TO SOME SPECIAL CONSTANTS

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Abstract. In this work, a new kernel function defined in the whole plane, including both the homogeneous and the non-homogeneous cases, and involving multiple parameters is constructed. By the method of weight coefficient and using some techniques of real analysis, a new Hilbert-type inequality with the newly constructed kernel function, as well as its equivalent Hardy-type inequality are established. The constant factors of the obtained inequalities are proved to be the best possible. Furthermore, assuming special values to the parameters, some interesting and special Hilbert-type inequalities with the constant factors involving some special constants, such as the Euler number, Bernoulli number and the Catalan constant are presented at the end of the paper.

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