SHARP INEQUALITIES FOR THE TOADER MEAN OF
ORDER −1 IN TERMS OF OTHER BIVARIATE MEANS

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Abstract. In the article, we present the best possible parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3, $\beta_4 \in \mathbb{R}$ such that the double inequalities

$$\frac{\alpha_1}{H(a,b)} + \frac{1-\alpha_1}{G(a,b)} < \frac{1}{T_{-1}(a,b)} < \frac{\beta_1}{H(a,b)} + \frac{1-\beta_1}{G(a,b)},$$

$$\frac{\alpha_2}{H(a,b)} + \frac{1-\alpha_2}{A(a,b)} < \frac{1}{T_{-1}(a,b)} < \frac{\beta_2}{H(a,b)} + \frac{1-\beta_2}{A(a,b)},$$

$$\frac{\alpha_3}{H(a,b)} + \frac{1-\alpha_3}{L(a,b)} < \frac{1}{T_{-1}(a,b)} < \frac{\beta_3}{H(a,b)} + \frac{1-\beta_3}{L(a,b)},$$

$$\frac{\alpha_4}{H(a,b)} + \frac{1-\alpha_4}{P(a,b)} < \frac{1}{T_{-1}(a,b)} < \frac{\beta_4}{H(a,b)} + \frac{1-\beta_4}{P(a,b)}$$

hold for all $a,b > 0$ with $a \neq b$, and provide several new bounds for the complete elliptic integral of the second kind, where $T_{-1}(a,b) = \left(\frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^{-1}\cos^2\theta + b^{-1}\sin^2\theta} d\theta \right)^2$ is the Toader mean of order −1, and $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$ and $A(a,b) = (a+b)/2$ are the harmonic, geometric, logarithmic, Seiffert and arithmetic means, respectively.

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U. S. Government Printing Office, Washington, 1964.

[2] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VOURINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[3] R. W. BARNARD, K. PEARCE, AND K. C. RICHARDS, *An inequality involving the generalized hypergeometric function and the arc length of an ellipse*, SIAM J. Math. Anal., 2000, **31** (3), 693–699.

[4] R. W. BARNARD, K. PEARCE, AND K. C. RICHARDS, *A monotonicity property involving $_3F_2$ and comparisons of the classical approximations of elliptical arc length*, SIAM J. Math. Anal., 2000, **32** (2), 403–419.

[5] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM*, John Wiley & Sons, New York, 1987.

[6] T. BURIĆ AND N. ELEZOVIĆ, *Asymptotic expansion of the arigthmetic-geometric mean and related inequalities*, J. Math. Inequal., 2015, **9** (4), 1181–1190.

[7] Y.-M. CHU, Y.-F. QIU, AND M.-K. WANG, *Hölder mean inequalities for the complete elliptic integrals*, Integral Transforms Spec. Funct., 2012, **23** (7), 521–527.

[8] Y.-M. CHU AND M.-K. WANG, *Optimal Lehmer mean bounds for the Toader mean*, Results Math., 2021, **61** (3–4), 223–229.

[9] Y.-M. CHU, M.-K. WANG, Y.-P. JIANG, AND S.-L. QIU, *Concavity of the complete elliptic integrals of the second kind with respect to Hölder means*, J. Math. Anal. Appl., 2012, **395** (2), 637–642.

[10] Y.-M. Chu, M.-K. Wang, and S.-L. Qiu, *Optimal combinations bounds of root-square and arithmetic means for Toader mean*, Proc. Indian Acad. Sci. Math. Sci., 2012, **122** (1), 41–51.

[11] Y.-M. Chu, M.-K. Wang, S.-L. Qiu, and Y.-P. Jiang, *Bounds for complete elliptic integrals of the second kind with applications*, Comput. Math. Appl., 2012, **63** (7), 1177–1184.

[12] H.-H. Chu, T.-H. Zhao, and Y.-M. Chu, *Sharp bounds for the Toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means*, Math. Slovaca, 2020, **70** (5), 1097–1112.

[13] N. Elezović and L. Mihoković, *Inequalities betwen reciprocals of means*, Sarajevo J. Math., 2015, **11 (24)** (2), 171–180.

[14] N. Elezović and L. Vukšić, *Neuman-Sándor mean, asymptotic expansions and related inequalities*, J. Math. Inequal., 2015, **9** (4), 1337–1348.

[15] Z.-Y. He, M.-K. Wang, Y.-P. Jiang, and Y.-M. Chu, *Bounds for the perimeter of an ellipse in terms of power means*, J. Math. Inequal., 2020, **14** (3), 887–899.

[16] T.-R. Huang, S.-Y. Tan, X.-Y. Ma, and Y.-M. Chu, *Monotonicity properties and bounds for the complete p-elliptic integrals*, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.

[17] X.-F. Huang, M.-K. Wang, H. Shao, Y.-F. Zhao, and Y.-M. Chu, *Monotonicity properties and bounds for the complete p-elliptic integrals*, AIMS Math., 2020, **5** (6), 7071–7086.

[18] W.-M. Qian and Y.-M. Chu, *Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters*, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.

[19] W.-M. Qian, Z.-Y. He, and Y.-M. Chu, *Approximation for the complete elliptic integral of the first kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2020, **114** (2), Article 57, 12 pages.

[20] W.-M. Qian, M.-K. Wang, H.-Z. Xu, and Y.-M. Chu, *Approximations for the complete elliptic integral of the second Kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, **115** (2), Article 88, 11 pages.

[21] S.-L. Qiu, X.-Y. Ma, and Y.-M. Chu, *Sharp Landen transformation inequalities for hypergeometric functions, with applications*, J. Math. Anal. Appl., 2019, **474** (2), 1306–1337.

[22] S.-L. Qiu, X.-Y. Ma, and Y.-M. Chu, *Extensions of quadratic transformation identities for hypergeometric functions*, Math. Inequal. Appl., 2020, **23** (4), 1391–1423.

[23] S.-L. Qiu, X.-Y. Ma, and Y.-M. Chu, *Transformation propertied of hypergeometric functions and their applications*, Comput. Methods Funct. Theory, 2021, https://doi.org/10.1007/s40315-021-00390-9.

[24] J.-M. Shen, Z.-H. Yang, W.-M. Qian, W. Zhang, and Y.-M. Chu, *Sharp rational bounds for the gamma function*, Math. Inequal. Appl., 2020, **23** (3), 843–853.

[25] S.-Y. Tan, T.-R. Huang, and Y.-M. Chu, *Functional inequalities for Gaussian hypergeometric function and generalized elliptic integral of the first kind*, Math. Slovaca, 2021, **71** (3), 667–682.

[26] Gh. Toader, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl., 1998, **218** (2), 358–368.

[27] L. Vukšić, *Seiffert means, asymptotic expansions and inequalities*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan, 2015, **19 (523)**, 129–142.

[28] M.-K. Wang and Y.-M. Chu, *Asymptotical bounds for complete elliptic integrals of the second kind*, J. Math. Anal. Appl., 2013, **402** (1), 119–126.

[29] M.-K. Wang and Y.-M. Chu, *Refinements of transformation inequalities for zero-balanced hypergeometric functions*, Acta Math. Sci., 2017, **37B** (3), 607–622.

[30] M.-K. Wang and Y.-M. Chu, *Landen inequalities for a class of hypergeometric functions with applications*, Math. Inequal. Appl., 2018, **21** (2), 521–537.

[31] M.-K. Wang, H.-H. Chu, and Y.-M. Chu, *Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals*, J. Math. Anal. Appl., 2019, **480** (2), Article ID 123388, 9 pages.

[32] M.-K. Wang, H.-H. Chu, and Y.-M. Chu, *On the approximation of some special functions in Ramanujan's generalized modular equation with signature 3*, Ramanujan J., 2021, **56** (1), 1–22.

[33] M.-K. Wang, Y.-M. Chu, and Y.-P. Jiang, *Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions*, Rocky Mountain J. Math., 2016, **46** (2), 679–691.

[34] M.-K. Wang, H.-H. Chu, Y.-M. Li, and Y.-M. Chu, *Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind*, Appl. Anal. Discrete Math., 2020, **14** (1), 255–271.

[35] M.-K. Wang, Y.-M. Chu, Y.-M. Li, and W. Zhang, *Asymptotic expansion and bounds for complete elliptic integrals*, Math. Inequal. Appl., 2020, **23** (3), 821–841.

[36] M.-K. WANG, Y.-M. CHU, S.-L. QIU, AND Y.-P. JIANG, *Convexity of the complete elliptic integrals of the first kind with respect to Hölder means*, J. Math. Anal. Appl., 2012, **388** (2), 1141–1146.

[37] M.-K. WANG, Y.-M. CHU, AND Y.-Q. SONG, *Asymptotical formulas for Gaussian and generalized hypergeometric functions*, Appl. Math. Comput., 2016, **276**, 44–60.

[38] M.-K. WANG, Y.-M. CHU, AND W. ZHANG, *Monotonicity and inequalities involving zero-balanced hypergeometric function*, Math. Inequal. Appl., 2019, **22** (2), 601–617.

[39] M.-K. WANG, Z.-Y. HE, AND Y.-M. CHU, *Sharp power mean inequalities for the generalized elliptic integral of the first kind*, Comput. Methods Funct. Theory, 2020, **20** (1), 111–124.

[40] M.-K. WANG, S.-L. QIU, Y.-M. CHU, AND Y.-P. JIANG, *Generalized Hersch-Pfluger distortion function and complete elliptic integrals*, J. Math. Anal. Appl., 2012, **385** (1), 221–229.

[41] G.-D. WANG, X.-H. ZHANG, AND Y.-M. CHU, *A power mean inequality involving the complete elliptic integrals*, Rocky Mountain J. Math., 2014, **44** (5), 1661–1667.

[42] H.-Z. XU, W.-M. QIAN, AND Y.-M. CHU, *Shapr bounds for the lemniscatic mean by the one-parameter geometric and quadratic means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2022, **116** (1), Article 21, 15 pages.

[43] Z.-H. YANG AND Y.-M. CHU, *A monotonicity property involving the generalized elliptic integral of the first kind*, Math. Inequal. Appl., 2017, **20** (3), 729–735.

[44] Z.-H. YANG, Y.-M. CHU, AND W. ZHANG, *High accuracy asymptotic bounds for the complete elliptic integral of the second kind*, Appl. Math. Comput., 2019, **348**, 552–564.

[45] Z.-H. YANG, W.-M. QIAN, AND Y.-M. CHU, *Monotonicity properties and bounds involving the complete elliptic integrals of the first kind*, Math. Inequal. Appl., 2018, **21** (4), 1185–1199.

[46] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU, AND W. ZHANG, *On rational bounds for the gamma function*, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.

[47] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU, AND W. ZHANG, *On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind*, J. Math. Anal. Appl., 2018, **462** (2), 1714–1726.

[48] Z.-H. YANG, W.-M. QIAN, W. ZHANG, AND Y.-M. CHU, *Notes on the complete elliptic integral of the first kind*, Math. Inequal. Appl., 2020, **23** (1), 77–93.

[49] Y.-Y. YANG, W.-M. QIAN, H.-W. ZHANG, AND Y.-M. CHU, *Shapr bounds for Toader-type means in terms of two-parameter means*, Acta Math. Sci., 2021, **41B** (3), 719–728.

[50] Z.-H. YANG, W. ZHANG, AND Y.-M. CHU, *Sharp Gautschi inequality for parameter $0 < p < 1$ with applications*, Math. Inequal. Appl., 2017, **20** (4), 1107–1120.

[51] T.-H. ZHAO, Z.-Y. HE, AND Y.-M. CHU, *On some refinements for inequalities involving zero-balanced hypergeometric function*, AIMS Math., 2020, **5** (6), 6479–6495.

[52] T.-H. ZHAO, Z.-Y. HE, AND Y.-M. CHU, *Sharp bounds for the weighted Hölder mean of the zero-balanced generalized complete elliptic integrals*, Comput. Methods Funct. Theory, 2021, **21** (3), 413–426.

[53] T.-H. ZHAO, W.-M. QIAN, AND Y.-M. CHU, *On approximating the arc lemniscate functions*, Indian J. Pure Appl. Math., 2021, https://doi.org/10.1007/s13226-021-00016-9.

[54] T.-H. ZHAO, Z.-H. SHEN, AND Y.-M. CHU, *Sharp power mean bounds for the lemniscate type means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, **115** (4), Article 175, 16 pages.

[55] T.-H. ZHAO, L. SHI, AND Y.-M. CHU, *Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2020, **114** (2), Article 96, 14 pages.

[56] T.-H. ZHAO, M.-K. WANG, AND Y.-M. CHU, *A sharp double inequality involving generalized complete elliptic integral of the first kind*, AIMS Math., 2020, **5** (5), 4512–4528.

[57] T.-H. ZHAO, M.-K. WANG, AND Y.-M. CHU, *Concavity and bounds involving generalized elliptic integral of the first kind*, J. Math. Inequal., 2021, **15** (2), 701–724.

[58] T.-H. ZHAO, M.-K. WANG, AND Y.-M. CHU, *Monotonicity and convexity involving generalized elliptic integral of the first kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, **115** (2), Article 46, 13 pages.

[59] T.-H. ZHAO, M.-K. WANG, W. ZHANG, AND Y.-M. CHU, *Quadratic transformation inequalities for Gaussian hypergeometric function*, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.

[60] T.-H. ZHAO, B.-C. ZHOU, M.-K. WANG, AND Y.-M. CHU, *On approximating the quasi-arithmetic mean*, J. Inequal. Appl., 2019, **2019**, Article 42, 12 pages.

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