OPTIMAL LEHMER MEAN BOUNDS FOR THE nTH POWER–TYPE TOADER MEANS OF n = −1, 1, 3

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Abstract. In the article, we prove that \( \lambda_1 = 0, \mu_1 = 5/8, \lambda_2 = -1/8, \mu_2 = 0, \lambda_3 = -1 \) and \( \mu_3 = -7/8 \) are the best possible parameters such that the double inequalities

\[
L_{\lambda_1}(a,b) < T_3(a,b) < L_{\mu_1}(a,b),
L_{\lambda_2}(a,b) < T_1(a,b) < L_{\mu_2}(a,b),
L_{\lambda_3}(a,b) < T_{-1}(a,b) < L_{\mu_3}(a,b)
\]

hold for \( a, b > 0 \) with \( a \neq b \), and provide new bounds for the complete elliptic integral of the second kind \( \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta \) on the interval \((0, 1)\), where \( L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p) \) is the \( p \)-th Lehmer mean and \( T_n(a,b) = \left( \frac{2}{\pi} \int_0^{\pi/2} a^n \cos^n \theta + b^n \sin^n \theta \, d\theta \right)^{2/n} \) is the \( n \)th power-type Toader mean.


Keywords and phrases: Lehmer mean, Gini mean, Toader mean, complete elliptic integrals.

REFERENCES

Gaussian hypergeometric function

Finally, we consider the Gaussian hypergeometric function, which is defined as
\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \]
where \((a)_n = a(a+1)\cdots(a+n-1)\) is the Pochhammer symbol.

The Gaussian hypergeometric function is a special function that arises in many
problems in mathematics and physics, such as in the solution of certain differential
equations and in the theory of special functions.

Some properties of the Gaussian hypergeometric function include:

1. **Series Representation**: The Gaussian hypergeometric function can be

represented as a power series, as shown above.

2. **Integral Representations**: There are several integral representations of the

Gaussian hypergeometric function, which can be useful in certain applications.

3. **Recurrence Relations**: The Gaussian hypergeometric function satisfies

certain recurrence relations, which can be used to simplify calculations.

4. **Integral Equations**: The Gaussian hypergeometric function can be

solved using integral equations, which can be useful in certain
problems.

5. **Analytic Continuation**: The Gaussian hypergeometric function can be

analytically continued to a larger domain, which can be useful in

many applications.

The Gaussian hypergeometric function is a very important special function
and has many applications in mathematics and physics. It is also

related to other special functions, such as the confluent hypergeometric function.

Differential Equations

The Gaussian hypergeometric function is a solution to the second-order

ordinary differential equation known as the hypergeometric differential

equation:

\[ (1-z) f''(z) + \left(c-(a+b+1)z\right)f'(z) - abf(z) = 0 \]

where \(f(z) = F(a, b; c; z)\).

This differential equation arises in many problems, such as in

the theory of special functions and in the solution of certain differential

equations.

Examples

To illustrate the Gaussian hypergeometric function, consider the following

example:

**Example**: Find the Gaussian hypergeometric function

\[ F(1, 2; 3; z) \]

**Solution**: Using the series representation, we have

\[ F(1, 2; 3; z) = \sum_{n=0}^{\infty} \frac{(1)_n (2)_n}{(3)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \frac{z^n}{n!} = 1 + z + z^2 + \cdots \]

This is a geometric series, which converges for \( |z| < 1 \).

Therefore, the Gaussian hypergeometric function

\[ F(1, 2; 3; z) = 1 + z + z^2 + \cdots \]

is a geometric series, which converges for \( |z| < 1 \).

Summary

The Gaussian hypergeometric function is a special function that arises in

many problems in mathematics and physics. It is defined as a series representation,

and it satisfies certain integral and recurrence relations. It is also

related to other special functions, such as the confluent hypergeometric function.

The differential equation associated with the Gaussian hypergeometric function

is known as the hypergeometric differential equation, and it arises in

many problems, such as in the theory of special functions and in the solution

of certain differential equations.

Historical Notes

The Gaussian hypergeometric function was first studied by

Karl Weierstrass in the 19th century. It has since been

studied by many mathematicians, and it continues to

be an active area of research in mathematics and physics.

Applications

The Gaussian hypergeometric function has many applications

in mathematics and physics, such as in the solution

of certain differential equations and in the theory

of special functions. It is also used in various fields

of science, such as in quantum mechanics and

statistical mechanics.

Further Reading

For more information on the Gaussian hypergeometric function,

see the following references:

- **A. Erdélyi et al., *Higher Transcendental Functions*, 1953.

These resources provide a comprehensive overview of the Gaussian

hypergeometric function and its applications.
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