GENERALIZED WEIGHTED SOBOLEV–MORREY ESTIMATES FOR HYPOELLIPTIC OPERATORS WITH DRIFT ON HOMOGENEOUS GROUPS

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Abstract. Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous group, Q be the homogeneous dimension of $\mathbb{G}$, $X_0, X_1, \ldots, X_m$ be left invariant real vector fields on $\mathbb{G}$ and satisfy Hörmander's rank condition on $\mathbb{R}^N$. Assume that $X_1, \ldots, X_m$ $(m \leqslant N-1)$ are homogeneous of degree one and X_0 is homogeneous of degree two with respect to the family of dilations $\left(\delta_\lambda\right)_{\lambda > 0}$. Consider the following hypoelliptic operator with drift on $\mathbb{G}$

$$\mathscr{L} = \sum_{i,j=1}^m a_{ij} X_i X_j + a_0 X_0,$$

where (a_{ij}) is a constant matrix satisfying the elliptic condition in $\mathbb{R}^m$ and $a_0 \neq 0$. In this paper, for this class of operators we obtain generalized weighted Sobolev-Morrey estimates by establishing boundedness of a large class of sublinear operators T_α, $\alpha \in [0, Q)$ generated by Calderón-Zygmund operators ($\alpha = 0$) and generated by fractional integral operator ($\alpha > 0$) on generalized weighted Morrey spaces and proving interpolation results in generalized weighted Sobolev-Morrey spaces on $\mathbb{G}$.

Mathematics subject classification (2020): Primary 35B65, 35H10, 35R03, 42B20, 42B35, 43A15, 43A80.

Keywords and phrases: Hypoelliptic operators with drift, homogeneous groups, fractional integral operator, singular integral operators, generalized weighted Morrey spaces, generalized weighted Sobolev-Morrey estimates.

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