

ON THE GENERALIZED POWER-TYPE TOADER MEAN

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Abstract. This paper deals with the so-called generalized power-type Toader mean which is defined by

$$T_n(a, b) = \left(\frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^n \cos^2 \theta + b^n \sin^2 \theta} d\theta \right)^{2/n}$$

for $a, b > 0$ with non-zero integer n . In this study, we establish the following chain of inequalities

$$H(a, b) < T_{-1}(a, b) < G(a, b) < T_1(a, b) < A(a, b) \\ < T_2(a, b) < Q(a, b) < T_3(a, b) < T_4(a, b) < C(a, b)$$

for all $a, b > 0$ with $a \neq b$, where $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$, $Q(a, b) = [(a^2 + b^2)/2]^{1/2}$ and $C(a, b) = (a^2 + b^2)/(a + b)$ are the harmonic, geometric, arithmetic, quadratic and contra-harmonic means, respectively. Further, we provide sharp bounds for $T_{-1}(a, b)$ and $T_4(a, b)$ in terms of bivariate means mentioned above. As applications, new bounds for complete elliptic integral of the second kind are established.

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