

INCREASING PROPERTY AND LOGARITHMIC CONVEXITY OF TWO FUNCTIONS INVOLVING DIRICHLET ETA FUNCTION

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Abstract. In the paper, with the help of an integral representation of the Dirichlet eta function and by means of a monotonicity rule for a ratio of two integrals with a parameter, the authors find increasing property and logarithmic convexity of two functions involving the gamma function, the extended binomial coefficient, and the Dirichlet eta function.

Mathematics subject classification (2020): Primary 11M06; Secondary 11B73, 11M41, 26A48, 26A51, 33B15.

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