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BOUNDARY SCHWARZ LEMMA FOR HARMONIC AND PLURIHARMONIC MAPPINGS IN THE UNIT BALL

HONG-PING LI AND MIODRAG MATELJEVIĆ*

Abstract. In this paper, we consider pluriharmonic and harmonic mappings f defined on the unit ball $\mathbb{B}^n$, $n \geqslant 2$, differentiable at a point a on the boundary of $\mathbb{B}^n$, and $f(\mathbb{B})$ satisfies some convexity hypothesis at $f(a)$. For those mappings f, we obtain versions of its boundary Schwarz lemma and the sharp estimate of the eigenvalue related to its Jacobian at a. In particular, Theorem **??** below, solves the corresponding extremal problems concerning the magnitude of the radial derivative of f at the direction a and improves the main estimates given in [7] and [12]. Moreover, we partly generalized the corresponding results given in [8] and [24].

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