

## A METHOD TO PROVE INEQUALITIES AND ITS APPLICATIONS

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*Abstract.* In this paper, we provide a method to prove an inequality. As applications of the method developed here, we establish Wilker and Huygens type inequalities for Gauss lemniscate functions, and present sharp inequalities between the inverse hyperbolic tangent and inverse sine functions. We also present sharp inequalities for trigonometric functions.

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