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COMPACTNESS OF EXTREMALS FOR SINGULAR TRUDINGER–MOSER INEQUALITIES ON THE WHOLE EUCLIDEAN SPACE

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Abstract. Let $W^{1,n}(\mathbb{R}^n)$ be the standard Sobolev space. For any $\tau>0$, $0<\beta<1$, Li and Yang [16] proved the existence of extremals for a singular Trudinger-Moser inequality. Namely, the supremum

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\,\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1}\int_{\mathbb{R}^n}\frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx$$

can be attained by some function $u_\beta\in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx=1$. Here $\Phi(n,t)=e^t-\sum_{j=0}^{n-2}t^j/j!$, and $\alpha_n=n\omega_{n-1}^{1/(n-1)}$ with ω_{n-1} being the surface area of the $(n-1)$-dimensional unit sphere. In this note, we consider the compactness of the function family $\{u_\beta\}_{0<\beta<1}$ and prove that up to a subsequence, u_β converges to some function u_0 in $C^1(\mathbb{R}^n)$ when $\beta\to 0$. Moreover, u_0 is an extremal function of the supremum

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\,\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1}\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx.$$

Let us explain the result in geometry. Denote $\omega_0(x)=\sum_{j=1}^n dx_j{}^2$ and $\omega_\beta(x)=|x|^{-2\beta}\omega_0(x)$ as the standard and conical metrics on $\mathbb{R}^n$. Then the extremal family $\{u_\beta\}_{0<\beta<1}$ of the following singular Trudinger-Moser functionals

$$\int_{\mathbb{R}^n}\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})dv_{\omega_\beta}$$

is compactness as $\beta\to 0$. This extends earlier result of Wang and Yang [33] and complements that of Li and Yang [16].

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