

NEW INEQUALITIES FOR QUOTIENTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

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Abstract. This paper deals with new inequalities involving the quotients

$$(\sin x)/(\sinh x), \quad (\cos x)/(\cosh x), \quad \text{and} \quad (\tan x)/(\tanh x).$$

The proofs are based on l'Hôpital's rule of monotonicity, series expansions using Bernoulli numbers, and some analytical techniques. Some of the obtained inequalities have a resemblance with Adamović-Mitrinović, Wilker and Cusa-Huygens type inequalities.

Mathematics subject classification (2020): 26D05, 26D07, 26D15, 33B10, 33B20.

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