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SHARP EXPONENTIAL TYPE INEQUALITIES FOR THE ARC LEMNISCATE SINE FUNCTION WITH APPLICATIONS

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Abstract. In this paper, by proving some monotonicity theorems of certain combinations of the arc lemniscate sine function and elementary functions, we obtain two classes of exponential type inequalities for the arc lemniscate sine function. As applications, sharp bounds for the lemniscatic mean in terms of the arithmetic, harmonic and geometric means are given, which extend some previously known results.

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Paper JMI-17-33

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