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ESTIMATING THE REMAINDER OF AN ALTERNATING p–SERIES USING HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, using hypergeometric functions, we provide sharp estimates of the remainder of the alternating p-series, $\sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n^p}$, where $p \geqslant 2$ is an integer. We show that the largest ρ and the largest σ such that the inequalities

$$\frac{1}{2(n+1)^p-\rho} \leqslant \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^p} \right| \leqslant \frac{1}{2n^p+\sigma},$$

hold for any integer $n \geqslant 1$ are

$$\rho(p) = 2^{p+1} - \frac{1}{1-(1-2^{1-p})\zeta(p)} \text{ and } \sigma(p) = \frac{1}{1-(1-2^{1-p})\zeta(p)} - 2,$$

where $\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}$, the Riemann zeta function.

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