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A THEOREM AND AN ALGORITHM INVOLVING MUIRHEAD'S INEQUALITY

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Abstract. Let $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ be column vectors, and $(\mathbf{u},\ \mathbf{v})$ be the inner product of vectors $\mathbf{u}$ and $\mathbf{v}$ on $\mathbb{R}^n$. Let $G\subset \mathrm{GL}(n,\mathbb{R})$ be a compact matrix group. For $A\in G$ and a continue function f on G, the integral $\int_G f(A)dA$ is the invariant integral of the compact group G. In this paper, we study the inequality

$$\forall \mathbf{x}\in\mathbb{R}^n \quad \int_G e^{(A\mathbf{a},\ \mathbf{x})}dA \geqslant \int_G e^{(A\mathbf{b},\ \mathbf{x})}dA.$$

We prove that the above inequality holds if and only if $\mathbf{b}\in\mathrm{Conv}(G\mathbf{a})$. This work follows a series of results, that is, Muirhead (1903), Hardy, Littlewood and Pòlya (1932), Rado (1952), Daykin (1971), Kimelfeld (1995) and Schulman (2009). Furthermore, We construct an determining algorithm when G is finite. Compared with other effective algorithms, this one is symbolic and easy to implement on computer.

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