

SHARP BOUNDS ON THE FOURTH-ORDER HERMITIAN TOEPLITZ DETERMINANT FOR STARLIKE FUNCTIONS OF ORDER $1/2$

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Abstract. In this paper, we prove the sharp bounds on the fourth-order Hermitian Toeplitz determinant for starlike functions of order $1/2$, which solves a conjecture posed by Cudna *et al.* [10] for the case $q = 4$.

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