NON–LINEAR CASAZZA–KALTON–CHRISTENSEN–VAN EIJNDHOVEN PERTURBATION WITH APPLICATIONS

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Abstract. Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $S : \mathscr{X} \to \mathscr{Y}$ be an invertible Lipschitz map. Let $T : \mathscr{X} \to \mathscr{Y}$ be a map and there exist $\lambda_1, \lambda_2 \in [0, 1)$ such that

$$\|Tx - Ty - (Sx - Sy)\| \leqslant \lambda_1 \|Sx - Sy\| + \lambda_2 \|Tx - Ty\|, \quad \forall x, y \in \mathscr{X}.$$

Then we prove that T is an invertible Lipschitz map. This is non-linear version of 26 years old Casazza-Kalton-Christensen-van Eijndhoven perturbation. It also a non-linear version of 29 years old Soderlind-Campanato perturbation and 3 years old Barbagallo-Ernst-Thera perturbation. We give applications to the theory of metric frames. The notion of Lipschitz atomic decomposition for Banach spaces is also introduced.

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