Journal of Mathematical Inequalities
Volume 18, Number 1 (2024), 315–328
doi:10.7153/jmi-2024-18-18

FURTHER IMPROVEMENTS FOR YOUNG'S INEQUALITIES ON THE ARITHMETIC, GEOMETRIC, AND HARMONIC MEAN

XIANGRUN YANG, CHANGSEN YANG AND HAIYING LI

Abstract. In this paper, we obtain some improvements and generalizations of Young's inequalities on the arithmetic, geometric, and harmonic mean. For example,

(1) If $0<a<b$, $\beta\geqslant 1$ and $0<v\leqslant\tau<1$, then

$$\frac{(a\nabla_v b)^\beta-(a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta-(a\sharp_\tau b)^\beta}\leqslant\frac{v(1-v)}{\tau(1-\tau)}.$$

(2) If $0<b<a$, $\beta\geqslant 1$ and $0<v\leqslant\tau<\frac{1}{2}$, then

$$\frac{(a\nabla_v b)^\beta-K(h,2)^{\beta v}(a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta-K(h,2)^{\beta\tau}(a\sharp_\tau b)^\beta}\geqslant\frac{v(1-v)}{\tau(1-\tau)};$$

(3) If $0<a<b$, $\beta\geqslant 1$ and $0<v\leqslant\tau<1$, then

$$\frac{(a\nabla_v b)^\beta-(a!_v b)^\beta}{(a\nabla_\tau b)^\beta-(a!_\tau b)^\beta}\leqslant\frac{(a\nabla_v b)-(a!_v b)}{(a\nabla_\tau b)-(a!_\tau b)}\leqslant\frac{v(1-v)}{\tau(1-\tau)}.$$

In addition, we obtain some new results for Young's inequality for operators.

Mathematics subject classification (2020): 26D07, 15A15, 15A42, 15A60.
Keywords and phrases: Young's inequality, Kantorovich constant, arithmetic mean, geometric mean, harmonic mean.

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Paper JMI-18-18

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