

ON A POSITIVITY PROPERTY OF THE REAL PART OF THE LOGARITHMIC DERIVATIVE OF THE RIEMANN ξ -FUNCTION

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Abstract. In this paper, we investigate the positivity of the real part of the logarithmic derivative of the Riemann ξ -function when $1/2 < \sigma < 1$ and t is sufficiently large. We consider explicit upper and lower bounds of $\Re \sum_{\rho} 1/(s - \rho)$, where the summation runs over the zeros of $\zeta(s)$ on the line $1/2 + it$. We also examine the positivity of $\Re \xi'/\xi(s)$ in the strip $1/2 < \sigma < 1$ assuming that there occur non-trivial zeros of $\zeta(s)$ off the critical line.

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