

## ON CONVEXITY AND POWER SERIES EXPANSION FOR LOGARITHM OF NORMALIZED TAIL OF POWER SERIES EXPANSION FOR SQUARE OF TANGENT

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*Abstract.* In the paper, the authors introduce the normalized tail of the Maclaurin power series expansion of the square of the tangent function, find out the logarithmic convexity of the normalized tail in light of the monotonicity rule for the ratio of two series, and expand the logarithm of the normalized tail into a Maclaurin power series with the help of a formula for higher order derivatives of the ratio of two differentiable functions.

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