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INEQUALITIES FOR POWER SERIES OF PRODUCT OF OPERATORS IN HILBERT SPACES WITH APPLICATIONS TO NUMERICAL RADIUS

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Abstract. Let H be a complex Hilbert space. We consider the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0,1,\ldots\}$. Suppose that this power series is convergent on the open disk $D(0,R) := \{z \in \mathbb{C} | z < R\}$. We define $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$, which has the same radius of convergence R. In this paper, we show among others that, if the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$ and A, B, $C, D \in B(H)$ with $\|AB\| < R$, then the following vector inequality holds

$$|\langle D^* ABf(AB)Cx, y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a(\|AB\|) \left\langle \left| |B|^{\alpha} C \right|^2 x, x \right\rangle^{1/2} \left\langle \left| |A^*|^{1-\alpha} D \right|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Application for norm and numerical radius inequalities for the composite operator $D^* ABf(AB)C$ are provided. Some examples for fundamental power series are also given.

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